Lecture 6

6.1. Hypothesis Tests

Summary

Hypothesis testing is a statistical procedure that uses sample data to determine whether a statement about the value of a population parameter should or should not be rejected. The hypotheses are two competing statements about a population parameter. One statement is called the null hypothesis, (H_0) and the other statement is called the alternative hypothesis (H_a). We provided guidelines for developing hypotheses for three situations frequently encountered in practice.

Whenever historical data or other information provides a basis for assuming that the population standard deviation is known, the hypothesis testing procedure is based on the standard normal distribution. Whenever σ is unknown, the sample standard deviation *s* is used to estimate σ and the hypothesis testing procedure is based on the *t*-distribution. In both cases, the quality of results depends on both the form of the population distribution and the sample size. If the population has a normal distribution, both hypothesis testing procedures are applicable, even with small sample sizes. If the population is not normally distributed, larger sample sizes are needed. In the case of hypothesis tests about a population proportion, the hypothesis testing procedure uses a test statistic based on the standard normal distribution.

In all cases, the value of the test statistic can be used to compute a p-value for the test. A p-value is a probability, computed using the test statistic, that measures the support (or lack of support) provided by the sample for the null hypothesis. If the *p*-value is less than or equal to the level of significance α, the null hypothesis can be rejected. Hypothesis testing conclusions can also be made by comparing the value of the test statistic to a critical value. For lower tail tests, the null hypothesis is rejected if the value of the test statistic is less than or equal to the critical value. For upper tail tests, the null hypothesis is rejected if the value of the test statistic is greater than or equal to the critical value. Two-tailed tests consist of two critical values: one in the lower tail of the sampling distribution and one in the upper tail. In this case, the null hypothesis is rejected if the value of the test statistic is less than or equal to the critical value in the lower tail or greater than or equal to the critical value in the upper tail.

Extensions of hypothesis testing procedures to include an analysis of the Type II error were also presented. We showed how to compute the probability of making a Type II error. We showed how to determine a sample size that will control for both the probability of making a Type I error and a Type II error.

Glossary

Null hypothesis. The hypothesis tentatively assumed true in the hypothesis testing procedure.

Alternative hypothesis. The hypothesis concluded to be true if the null hypothesis is rejected

Type I error. The error of rejecting H_o when it is true.

Type II error. The error of accepting H_o when it is false.

Level of significance. The probability of making a Type I error when the null hypothesis is true as an equality.

One-tailed test. A hypothesis test in which rejection of the null hypothesis occurs for values of the test statistic in one tail of its sampling distribution.

Test statistic. A statistic whose value helps determine whether a null hypothesis can be rejected

p-value. A probability, computed using the test statistic, that measures the support (or lack of support) provided by the sample for the null hypothesis. For a lower tail test, the p-value is the probability of obtaining a value for the test statistic at least as small as that provided by the sample. For an upper tail test, the p-value is the probability of obtaining a value for the test statistic at least as large as that provided by the sample. For a two-tailed test, the p-value is the probability of obtaining a value for the test statistic at least as unlikely as that provided by the sample.

Critical value. A value that is compared with the test statistic to determine whether H_o should be rejected.

Two-tailed test. A hypothesis test in which rejection of the null hypothesis occurs for values of the test statistic in either tail of its sampling distribution. **Power** The probability of correctly rejecting H_o when it is false.

Power curve. A graph of the probability of rejecting H_o for all possible values of the population parameter not satisfying the null hypothesis. The power curve provides the probability of correctly rejecting the null hypothesis.

Key formulas

Test Statistic for Hypothesis Tests About a Population Mean: σ Known

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \tag{6.1}$$

Test Statistic for Hypothesis Tests About a Population Mean: σ Unknown

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \tag{6.2}$$

Test Statistic for Hypothesis Tests About a Population Proportion

$$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \tag{6.3}$$

Sample Size for a One-Tailed Hypothesis Test About a Population Mean

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_a)^2} \tag{6.4}$$

in a two-tailed test, replace z_α with $z_{\alpha/2}$-

Lecture supplementary material

Table 1. Errors and correct conclusions in hypothesis testing

Conclusion	Population condition		
		H_0 True	**H_a true**
	Accept H_0	Correct conclusion	Type II Error
	Reject H_0	Type I Error	Correct conclusion

Table 2. Summary of hypothesis test about a population mean: **σ known case**

	Lower Tail Test	**Upper Tail Test**	**Two-Tailed Test**
Hypotheses	$H_0: \mu \ge \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \le \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \ne \mu_0$
Test Statistic	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
Rejection Rule: p-Value Approach	Reject H_0 if p-value$\le\alpha$	Reject H_0 if p-value$\le\alpha$	Reject H_0 if p-value$\le\alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \le -z_\alpha$	Reject H_0 if $z \ge z_\alpha$	Reject H_0 if $z \le -z_{\alpha/2}$ or if $z \ge z_{\alpha/2}$

Table 3. Summary of hypothesis test about a population mean: **σ unknown case**

	Lower Tail Test	**Upper Tail Test**	**Two-Tailed Test**
Hypotheses	$H_0: \mu \ge \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \le \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \ne \mu_0$
Test Statistic	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$
Rejection Rule: p-Value Approach	Reject H_0 if p-value$\le\alpha$	Reject H_0 if p-value$\le\alpha$	Reject H_0 if p-value$\le\alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $t \le -t_\alpha$	Reject H_0 if $t \ge t_\alpha$	Reject H_0 if $t \le -t_{\alpha/2}$ or if $t \ge t_{\alpha/2}$

Table 4. Summary of hypothesis test about a population proportion

	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: p \geq p_0$ $H_a: p < p_0$	$H_0: p \leq p_0$ $H_a: p > p_0$	$H_0: p = p_0$ $H_a: p \neq p_0$
Test Statistic	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$	$z = \frac{\bar{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
Rejection Rule: p-Value Approach	Reject H_0 if p-value$\leq\alpha$	Reject H_0 if p-value$\leq\alpha$	Reject H_0 if p-value$\leq\alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \leq -z_\alpha$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

6.2. Statistical Inference about Means and Proportions with Two Populations

Summary

In this lecture we discussed procedures for developing interval estimates and conducting hypothesis tests involving two populations. First, we showed how to make inferences about the difference between two population means when independent simple random samples are selected. We first considered the case where the population standard deviations, σ_1 and σ_2, could be assumed known. The standard normal distribution z was used to develop the interval estimate and served as the test statistic for hypothesis tests. We then considered the case where the population standard deviations were unknown and estimated by the sample standard deviations s_1 and s_2. In this case, the t distribution was used to develop the interval estimate and served as the test statistic for hypothesis tests.

Inferences about the difference between two population means were then discussed for the matched sample design. In the matched sample design each element provides a pair of data values, one from each population. The difference between the paired data values is then used in the statistical analysis. The matched sample design is generally preferred to the independent sample design because the matched-sample procedure often improves the precision of the estimate.

Finally, interval estimation and hypothesis testing about the difference between two population proportions were discussed. Statistical procedures for analyzing the difference between two population proportions are similar to the procedures for analyzing the difference between two population means.

Glossary

Independent samples Samples selected from two populations in such a way that the elements making up one sample are chosen independently of the elements making up the other sample.

Matched samples Samples in which each data value of one sample is matched with a corresponding data value of the other sample.

Pooled estimator of *p* An estimator of a population proportion obtained by computing a weighted average of the point estimators obtained from two independent samples.

Key formulas

Point Estimator of the Difference Between Two Population Means

$$\bar{x}_1 - \bar{x}_2 \tag{6.5}$$

Standard Error of $\bar{x}_1 - \bar{x}_2$

$$\sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \tag{6.6}$$

Internal Estimate of the Difference Between Iwo Population Means: σ_1 and σ_2 known

$$\bar{x}_1 - \bar{x}_2 \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \tag{6.7}$$

Test Statistic for Hypothesis Tests About $\mu_1 - \mu_2$: σ_1 and σ_2 known

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \tag{6.8}$$

Interval Estimate of the Difference Between Two Population Means: σ_1 and σ_2 unknown

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \tag{6.9}$$

Degrees of freedom for the t Distribution Using Two Independent Random Sample

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{1}{n_1 - 1}\left(\frac{s_1^2}{n_1}\right)^2 + \frac{1}{n_2 - 1}\left(\frac{s_2^2}{n_2}\right)^2} \tag{6.10}$$

Test Statistic for Hypothesis Tests about $\mu_1 - \mu_2$: σ_1 and σ_2 Unknown

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - D_0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \tag{6.11}$$

Test Statistic for Hypothesis Tests Involving Matched Samples

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \tag{6.12}$$

Point Estimate of the Difference Between Two Population Proportions

$$\bar{p}_1 - \bar{p}_2 \tag{6.13}$$

Standard Error of $p_1 - p_2$

$$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}} \tag{6.14}$$

Interval Estimate of the Difference Between Two Population Proportions

$$\bar{p}_1 - \bar{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\bar{p}_1(1 - \bar{p}_1)}{n_1} + \frac{\bar{p}_2(1 - \bar{p}_2)}{n_2}} \tag{6.15}$$

Standard Error of $p_1 - p_2$ When $p_1 = p_2 = p$

$$\sigma_{\bar{p}_1 - \bar{p}_2} = \sqrt{p(1 - p)\left(\frac{p(1 - p)}{n_1}\right)} \tag{6.16}$$

Pooled Estimator of p When $p_1 = p_2 = p$

$$\bar{p} = \frac{n_1\bar{p}_1 + n_2\bar{p}_2}{n_1 + n_2} \tag{6.17}$$

Test Statistic for Hypothesis Tests About $p_1 - p_2$

$$z = \frac{(\bar{p}_1 - \bar{p}_2)}{\sqrt{\bar{p}(1 - \bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \tag{6.18}$$